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Small-Mass Behavior of Quantum Gibbs States for Lattice Models with Unbounded Spins

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We construct the distribution of the infinite-dimensional Markov process associated with a finite-temperature Gibbs state for a quantum mechanical anharmonic crystal. The corresponding state is constructed via a cluster expansion technique for an arbitrary fixed temperature and, correspondingly, small enough masses of particles.

KEY WORDS: Quantum Gibbs state; lattice model; unbounded spin; small mass; cluster expansion.

1. INTRODUCTION

The small mass dependence (or "strong quantumness") of physical quantum systems has been investigated in recent years from different points of view. The suppression of the long-range order by strong quantum fluctuations in such systems was experimentally observed (see, e.g., Tibballs *et al.*[1]) and discussed long ago from the physical point of view (see, e.g., Schneider *et al.*[2], or the book (ref. 3, Chapter 2.5.4.3)). A rigorous treatment of this phenomenon was given by Verbeure and Zagrebnov.[4] The suppression not only of the long-range order but also of any critical anomalies was proved by Albeverio, Kondratiev, and Kozitsky.[5]

This paper is related to a previous article by Minlos, Verbeure, and Zagrebnov[6] in which the properties of limit Gibbs states for an infinite

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system of interacting anharmonic oscillators on a ν-dimensional lattice $\mathbb{Z}^\nu$ were studied in the limit of small masses of the particles.

We study the finite temperature state of quantum anharmonic crystals. As usual, the construction of such a state by a Feynman–Kac formula technique reduces to the construction of some measure $\mu(\cdot)$ on the space of periodic trajectories (see refs. 7 and 8 for details) $\Omega = \{\omega_j(\tau),\ j \in \mathbb{Z}^\nu, \tau \in [0, \beta],\ \omega_j(0) = \omega_j(\beta)\}$, where β^{-1} is the temperature of the crystal system. This measure is the distribution of some Markovian process with values in $(\mathbb{R})^{\mathbb{Z}^\nu}$ (see ref. 6).

We construct this measure with the help of some variant of the cluster expansion technique which goes back to the works of Brydges and Federbush[9, 10] and was developed for the system under consideration by Kondratiev and Rebenko.[11] As we mentioned above, the present work is close to the results of the paper,[6] in which finite-temperature and ground states were constructed (also with the assumption of small mass) with the help of another cluster expansion which required a partition of every trajectory into proper pieces. A control of the convergence for the cluster expansion in ref. 6 is based on a delicate combination of combinatorial and probabilistic estimations with an additional use of WBK asymptotics for one-particle Hamiltonians. In contrast to the quite complicated technique of ref. 6, our cluster expansion estimates reduce to the estimation of the moments of trajectories $\langle \omega_{j_k}(\tau_1) \cdots \omega_{j_k}(\tau_m) \rangle$ at the same sites and consequently, by Gaussian upper-bound inequalities, to detailed estimations of two-point moments $\langle \omega_{j_k}(\tau_1)\, \omega_{j_k}(\tau_2) \rangle$. The latter have an explicit dependence on the mass parameter which creates the desired quantum effect. We consider this method of estimation as a new input in the study of this type of models; its implementation constitutes the main technical achievement in this paper.

In the present paper we consider the Brydges–Federbush type expansion. In ref. 6 all necessary constructions, allowing to construct quantum states on the quasilocal algebra of states were made using a cluster expansion for the measure $\mu(\cdot)$, we mention here only the most important details.

The brief contents of this paper is the following. In Section 1 we define our system and formulate the main result. In Section 2 we construct the cluster expansion and give a brief proof of the main theorem, and in Section 3 we provide all necessary estimates.

2. DESCRIPTION OF THE SYSTEM AND MAIN RESULT

We consider a one-component continuous spin system on a ν-dimensional cubic lattice $\mathbb{Z}^\nu$. With each site $j \in \mathbb{Z}^\nu$ a one-particle physical Hilbert space $L^2(\mathbb{R}^1, dq)$ is associated, where dq is the Lebesgue measure on $\mathbb{R}^1$. Then

$$\mathscr{H}_\Lambda = \bigotimes_{j \in \Lambda} L^2(\mathbb{R}^1, dq_j) = L^2((\mathbb{R}^1)^{|\Lambda|}, dq_\Lambda)$$

$$dq_\Lambda = \prod_{j \in \Lambda} dq_j$$

is the Hilbert space related to some given bounded set $\Lambda \subset \mathbb{Z}^\nu$, $|\Lambda| < \infty$.

For every finite set $\Lambda \subset \mathbb{Z}^\nu$ we consider the Hamiltonian ($\hbar = 1$) in $\mathscr{H}_\Lambda$:

$$H_\Lambda = -\frac{1}{2m} \sum_{j \in \Lambda} \frac{\partial^2}{\partial q_j^2} + \sum_{j \in \Lambda} V(q_j) + \frac{J}{2} \sum_{\langle i, j \rangle \subset \Lambda} (q_i - q_j)^2 \tag{2.1}$$

where m is the mass of the particles, the sum being extended over all pairs $\langle i, j \rangle \subset \Lambda$ for which $|i-j| = 1$ and $J > 0$.[5]

We consider a one-particle potential $V(q)$ in (2.1) of the following form

$$V(q) = v(q^2) = \sum_{p=1}^{s} a_{2p} q^{2p}, \qquad a_{2s} > 0, \qquad s > 2, \qquad s \in \mathbb{N} \tag{2.2}$$

An additional assumption on $V(q)$ is that $v(\cdot)$ be convex on $\mathbb{R}^1_+$.

Let $\mathscr{L}(\mathscr{H}_\Lambda)$ be the algebra of bounded operators in $\mathscr{H}_\Lambda$. Let us consider the temperature Gibbs state on $\mathscr{L}(\mathscr{H}_\Lambda)$:

$$\rho_\Lambda^\beta(A) = \frac{\operatorname{Tr}(A e^{-\beta H_\Lambda})}{Z_\beta(\Lambda)} \tag{2.3}$$

where $Z_\beta(\Lambda) = \operatorname{Tr} e^{-\beta H_\Lambda}$ and $A \in \mathscr{L}(\mathscr{H}_\Lambda)$.

Note, that all algebras $\mathscr{L}(\mathscr{H}_\Lambda)$ (for different Λ) are naturally embedded in each other

$$\mathscr{L}(\mathscr{H}_\Lambda) \subset \mathscr{L}(\mathscr{H}_{\Lambda'}), \qquad \Lambda \subset \Lambda'$$

Using this fact we can define the inductive limit

$$\mathfrak{A}_0 = \lim_{\Lambda \nearrow \mathbb{Z}^\nu} \mathscr{L}(\mathscr{H}_\Lambda)$$

[5] We consider a nearest-neighbours interaction just for simplicity.

which is called the algebra of local observables. The closure of this algebra in norm forms the quasilocal algebra

$$\mathfrak{A} = \bar{\mathfrak{A}}_0$$

The main result of this paper is the following theorem.

Theorem 2.1. For the system of quantum particles with interactions (2.1), (2.2) and for any β one can find some $m_0 = m_0(\beta)$, such that for any $0 < m \leqslant m_0$ the limit

$$\lim_{\Lambda \nearrow \mathbb{Z}^\nu} \rho_\Lambda^\beta(A) = \rho^\beta(A), \qquad A \in \mathfrak{A}_0$$

exists. $\rho^\beta(A)$ gives a state on the algebra $\mathfrak{A}_0$, which can be continuously extended to the algebra $\mathfrak{A}$.

Using the results of paper[6] it is sufficient to prove this theorem for some sub-algebra $\mathfrak{A}^\tau \subset \mathfrak{A}$ of local operators, which we describe below.

The main technical tools are the Feynman–Kac formula and the representation of the states (2.3) on this algebra by functional integrals with respect to the measure $\mu(\cdot)$ (see refs. 7 and 8 for details).

But before we rewrite our system in the language of a functional integral technique, we make the following standard change of variables[5, 6]:

$$q_j = \alpha x_j, \qquad \alpha = m^{-1/2(s+1)} \tag{2.4}$$

The change of variables (2.4) induces the unitary map:

$$U\colon \mathscr{H}_\Lambda \to \mathscr{H}_\Lambda$$

$$(Uf)(x_\Lambda) = \alpha^{|\Lambda|/2} f((\alpha x)_\Lambda), \qquad (\alpha x)_\Lambda = \{\alpha x_j,\ j \in \Lambda\}$$

It is easy to check that

$$U H_\Lambda U^{-1} = m^{-s/(s+1)} \hat{H}_\Lambda$$

where

$$\hat{H}_\Lambda = \sum_{j \in \Lambda} \hat{h}_j + \hat{W}_\Lambda \tag{2.5}$$

$$\hat{h}_j = -\frac{1}{2} \frac{\partial^2}{\partial x_j^2} + \hat{V}(x_j) \tag{2.6}$$

$$\hat{V}(x_j) = a_{2s} x_j^{2s} + \sum_{p=1}^{s-1} a_{2p} m^{(s-p)/(s+1)} x_j^{2p} \tag{2.7}$$

$$\hat{W}_\Lambda = \frac{1}{2} J m^{(s-1)/(s+1)} \sum_{\langle i,j\rangle \subset \Lambda} (x_i - x_j)^2$$

Note that all powers of m in $\hat{V}$ are positive ($(s-p)/(s+1) > 0$ for $p = 1,..., s-1$). We also define

$$\hat{\beta} = \beta m^{-s/(s+1)} \tag{2.8}$$

which yields

$$U\beta h_\Lambda U^{-1} = \hat{\beta}\hat{H}_\Lambda$$

Then

$$\rho_\Lambda^\beta(A) = \hat{\rho}_\Lambda^{\hat{\beta}}(\hat{A}), \qquad \hat{A} = UAU^{-1}$$

where

$$\hat{\rho}_\Lambda^{\hat{\beta}}(\cdot) = \hat{Z}_{\hat{\beta}}^{-1}(\Lambda) \operatorname{Tr}(\cdot \exp(-\hat{\beta}\hat{H}_\Lambda)) \tag{2.3'}$$

Later on we consider the state $\hat{\rho}_\Lambda^{\hat{\beta}}$ and its limit as $\Lambda \nearrow \mathbb{Z}^\nu$, and so omit the "hat" $\hat{}$ in the definitions of $\hat{\rho}$ and $\hat{Z}(\Lambda)$ (retaining it for $\hat{\beta}$).

Now, for given $\hat{\beta}$ and $\Lambda \subset \mathbb{Z}^\nu$, we consider the space of periodic trajectories $(\Omega_{\hat{\beta},\Lambda}, \Sigma_{\hat{\beta},\Lambda})$, where

$$\Omega_{\hat{\beta},\Lambda} = \{\omega_\Lambda(\cdot) \mid \omega_\Lambda : S_{\hat{\beta}} \to \mathbb{R}^\Lambda\}$$

$$\omega_\Lambda(\cdot) = \{\omega_j(\cdot), j \in \Lambda \mid \omega_j \in \Omega, \Omega := C(S_{\hat{\beta}} \to \mathbb{R})\}$$

and $\Sigma_{\hat{\beta},\Lambda}$ is the standard σ-algebra of $\Omega_{\hat{\beta},\Lambda}$-subsets generated by Borel cylinder subsets.[7,8] Then we define the "free" measure on $\Omega_{\hat{\beta},\Lambda}$ by the formula

$$d\mu_{\hat{\beta},\Lambda}^0 = d\mu_{\hat{\beta}}^0(\omega_\Lambda) = \bigotimes_{j\in\Lambda} d\mu_{\hat{\beta}}^0(\omega_j)$$

where $d\mu_{\hat{\beta}}^0(\omega_\Lambda)$ is determined by the one-particle Hamiltonian. Sometimes it is convenient to have in mind the following heuristic representation

$$\begin{aligned} d\mu_{\hat{\beta}}^0(\omega_\Lambda) &= \frac{1}{Z(\hat{\beta})} \exp\left\{-\frac{1}{2}\int_{S_{\hat{\beta}}} \dot{\omega}_j^2(\tau)\, d\tau - \int_{S_{\hat{\beta}}} \hat{V}(\omega_j(\tau))\, d\tau\right\} d\omega_j \\ d\omega_j &= \prod_{\tau \in S_{\hat{\beta}}} d\omega_j(\tau) \end{aligned} \tag{2.9}$$

For each finite $\Lambda \subset \mathbb{Z}^\nu$ we define the Gibbsian modification $d\mu_\beta(\omega_\Lambda)$ of the "free" measure $d\mu_\beta^0(\omega_\Lambda)$ by the Radon–Nikodym derivative

$$\frac{d\mu_\beta(\omega_\Lambda)}{d\mu_\beta^0(\omega_\Lambda)} = \frac{1}{Z_\beta(\Lambda)} \exp\left\{-\frac{1}{2} Jm^{(s-1)/(s+1)} \sum_{\langle i,j\rangle \subset \Lambda} \int_{S_\beta} (\omega_i(\tau) - \omega_j(\tau))^2 \, d\tau\right\} \tag{2.10}$$

Now for every bounded function $\mathscr{A}(x_\Lambda)$ on $\mathbb{R}^\Lambda$ we consider the bounded operator A_0 in $\mathscr{H}_\Lambda$

$$(A_0 f)(x_\Lambda) = \mathscr{A}(x_\Lambda)\, f(x_\Lambda) \tag{2.11}$$

and for any $t > 0$ we define

$$A_t = e^{-tH_\Lambda} A_0 e^{tH_\Lambda} \tag{2.12}$$

Then for every set of bounded functions $\mathscr{A}^{(0)},..., \mathscr{A}^{(n)}$ and increasing sequence of moments $0 = t_0 < t_1 < \cdots < t_n < \hat{\beta}$, we define the operator

$$A = \prod_{0 \leqslant l \leqslant n} A_{t_l}^{(l)} = A^{(0)} e^{-t_1 H_\Lambda} A^{(1)} e^{-(t_2 - t_1) H_\Lambda} \cdots e^{-(t_n - t_{n-1}) H_\Lambda} A^{(n)} e^{t_n H_\Lambda} \tag{2.13}$$

and write for this operator our state (2.3′)

$$\rho_\Lambda^{\hat{\beta}}(A) = \frac{1}{Z_\beta(\Lambda)} \times \operatorname{Tr}(A_0^{(0)} e^{-t_1 H_\Lambda} A_0^{(1)} e^{-(t_2 - t_1) H_\Lambda} \cdots e^{-(t_n - t_{n-1}) H_\Lambda} A_0^{(n)} e^{-(\beta - t_n) H_\Lambda}) \tag{2.14}$$

This definition is correct because every operator $A_0^{(k)} e^{-(t_{k+1} - t_k) H_\Lambda}$, $k = 0, 1,..., n$, with $t_0 = 0$ and $t_{n+1} = \hat{\beta}$, is of trace class. Then the following formula is true

$$\rho_\Lambda^{\hat{\beta}}\left(\prod_{l=0}^{n} A_{t_l}^{(l)}\right) = \left\langle \prod_{l=0}^{n} \mathscr{A}^{(l)}(\cdot) \right\rangle_{\mu_{\hat{\beta},\Lambda}} = \int_{\Omega_{\hat{\beta},\Lambda}} \prod_{l=0}^{n} \mathscr{A}(\omega_\Lambda(t_l))\, d\mu_{\hat{\beta},\Lambda} \tag{2.15}$$

So, as we mentioned above, using ref. 6 we can reformulate Theorem 2.1 for the states (2.12) as follows:

Theorem 2.2. For the system of quantum particles with interactions (2.5)–(2.7) and any fixed β there exists a sufficiently small value $m_0(\beta)$ of the mass such that for all $0 < m \leqslant m_0$ the weak limit of the measures

$$\lim_{\Lambda \nearrow \mathbb{Z}^\nu} \mu_{\beta, \Lambda} = \mu_\beta$$

exists. μ_β is thus the limit Gibbs measure on the space $\Omega_{\beta, \Lambda}$.

To prove this theorem we are going to apply a cluster expansion procedure to the measure μ_β. This is the contents of the following section.

Remark 2.1. For the particular case, where we choose just one time $t_0 = 0$ and corresponding function $\mathscr{A}^{(0)}$ we have from (2.13)

$$\rho_\Lambda^\beta(A) = \langle \mathscr{A}^{(0)}(\omega_\Lambda(0)) \rangle_{\mu_\beta, \Lambda}$$

and thus, we define the state $\rho_\Lambda^\beta(\cdot)$ on the commutative subalgebra $\mathfrak{A}_\Lambda^{com} \subset \mathfrak{A}_0$ of the operator (2.11). Using formula (2.15) we can extend the class of the observables for which the state $\rho_\Lambda^\beta(\cdot)$ is defined by the measure $\mu_\Lambda^\beta(\cdot)$, but nevertheless this class remains very small. Therefore, the construction of the state $\rho_\Lambda^\beta(\cdot)$ on the whole algebra $\mathfrak{A}(\mathscr{H}_\Lambda)$ (as well as of the limit state ρ^β on the local algebra $\mathfrak{A}_0$) can be carried out by a slight generalization of our construction, as discussed in ref. 6 (and already mentioned in the Introduction).

Remark 2.2. Theorem 2.1 proves the existence of the limit measure on the space $\Omega_{\beta, \Lambda}$ and its uniqueness for the case where the conditional measures (for any $\Lambda \subset \mathbb{Z}^\nu$, $|\Lambda| < \infty$) are given by equations (2.10), which corresponds (in DLR-language) to empty boundary conditions. But the uniqueness of the limit measure for arbitrary boundary conditions (or, at least, some class of boundary conditions) is still an open problem. Along the lines of the works by Albeverio, Høegh-Krohn, and Zegarlinski[12] and Albeverio, Kondratiev, Tsikalenko, and Röckner[13] this could be proved from the convergence of the cluster expansion.

3. CLUSTER EXPANSION. PROOF OF THEOREM 2.2

The cluster expansion which we are going to use was constructed in ref. 11 using ideas of refs. 9 and 10.

The starting point of the cluster expansion for a Gibbs measure is the cluster expansion for the corresponding partition function

$$Z_{\beta}(\Lambda)=\sum_{B} K_B Z_{\beta}(\Lambda\backslash B)=\sum_{\{B_1,\dots,B_s\}}\prod_{l} K_{B_l},\qquad \bigcup_{l} B_l=\Lambda \tag{3.1}$$

where the sum is extended over all unordered collections of sets of mutually disjoint subsets $B_l\subset\Lambda$ giving a partition of Λ. In addition, in the content of Brydges–Federbush type of cluster expansion (see ref. 11), the expressions for K_B have the following form: for $|b|=1$, $B=\{j\}\in\mathbb{Z}^{\nu}$

$$K_{\{j\}}=\int \exp\left\{-\lambda\nu\int_{S_{\beta}}\omega_j^2\,d\tau\right\}d\mu^0_{\beta,j}(\omega_j)=\int x_{\{j\}}(\omega_j)\,d\mu^0_{\beta,j}(\omega_j) \tag{3.2}$$

where $\lambda=Jm^{(s-1)/(s+1)}$ and the integral is defined with respect to the one-site measure $d\mu^0_{\beta,j}(\omega_j)$ (see (2.9)); for B with $|B|>1$, we can take in every such set some point $k_1=k_1(B)$ (for example the smallest one in the lexicographic ordering of the lattice $\mathbb{Z}^{\nu}$) and then consider the collection of indices of the lattice sites of B (i.e., the enumeration of points of B) and their simultaneous couplings:

$$\eta\colon \{k_1=k_1(B), k_2,\dots, k_{|b|}\}\mapsto\{(k_1,k_2),(k_{\eta(3)},k_3),\dots,(k_{\eta(n)},k_n)\}$$

such that $\eta(l)<l$, $l=2,\dots,n$, $n=|b|$, and $|k_l-k_{\eta(l)}|=1$. Note that the latter condition implies that B is a 1-connected subset of $\mathbb{Z}^{\nu}$ (i.e., a connected subgraph of $\mathbb{Z}^{\nu}$, where the edges are neighboring pairs $(k_1,k_2)\subset\mathbb{Z}^{\nu}$, $|k_1-k_2|=1$). Then, for K_B with 1-connected B ($|B|>1$), we get (see refs. 9–11 for details):

$$\begin{aligned}
x_B(\omega_B)=\lambda^{n-1}\sum_{\eta}\int_0^1(ds)^{n-1}\,h_{\eta}(s)\prod_{l=2}^{n}\int_0^{\beta}\omega_{k_{\eta(l)}}(\tau)\,\omega_{k_l}(\tau)\,d\tau\\
\times\exp\Bigg\{-\nu\lambda\sum_{j\in b}\int_0^{\beta}\omega_j^2(\tau)\,d\tau\\
+\lambda\sum_{1\leqslant l<m\leqslant n}s_l\cdots s_{m-1}\int_0^{\beta}\omega_{k_l}(\tau)\,\omega_{k_m}(\tau)\,d\tau\Bigg\}
\end{aligned} \tag{3.3}$$

$$K_B=\int x_B(\omega_B)\,d\mu^0_{\beta,B}(\omega_B)$$

where the integral is defined with respect to the product-measure

$$\mu^0_{\beta,B}(\omega_B)=\prod_{k\in B}\mu^0_{\beta,k}(\omega_k)$$

And finally

$$h_\eta(s) = \prod_{2 \leqslant m \leqslant n} (s_{\eta(m)} s_{\eta(m)+1} \cdots s_{m-2}) \tag{3.4}$$

$$\eta(m) \leqslant m-2, \qquad s = (s_1, ..., s_{n-1}), \qquad 0 \leqslant s_j \leqslant 1 \tag{3.5}$$

The product in (3.4) is equal to 1 if $\eta(m) > m-2$.

Using the same method one can construct a similar expression for averages $\langle \mathscr{A}_{\mathscr{D}}(\cdot) \rangle_{\mu_{\beta, \Lambda}}$, where $\mathscr{A}_{\mathscr{D}} = \mathscr{A}_{\mathscr{D}}(\omega_{\mathscr{D}})$, $\mathscr{D} \subset \Lambda$ is local function which depends on the trajectories $\{\omega_j(\tau), j \in \mathscr{D}\}$:

$$\langle \mathscr{A}_{\mathscr{D}}(\cdot) \rangle_{\mu_{\beta, \Lambda}} = \sum_{B \subset \Lambda, \mathscr{D} \subset B} K_B(\mathscr{A}_{\mathscr{D}}) \frac{Z_\beta(\Lambda \backslash B)}{Z_\beta(\Lambda)} \tag{3.6}$$

where $K_B(\mathscr{A}_{\mathscr{D}})$ is defined by the formula

$$K_B(\mathscr{A}_{\mathscr{D}}) = \int \Bigg(\sum_{\substack{\{B_1, ..., B_m\} \\ \bigcup B_i = B,\, B_i \cap B_j = \varnothing\; i \neq j \\ B_1 \cap \mathscr{D} \neq \varnothing}} \prod_{i=1}^{m} x_{B_i}(\omega_{B_i}) \Bigg) d\mu_B^0(\omega_B), \qquad \mathscr{D} \subseteq B \tag{3.7}$$

Here summation goes over all partitions of $B = B_1 \cup \cdots \cup B_s$ such that $B_1 \cap \mathscr{D} \neq \varnothing$.

To prove Theorem 2.2 we use (3.5) and the following two lemmas.

Lemma 3.1. For a given temperature there exists a sufficiently small value of the mass m_0, such that for all $0 < m < m_0$ there exists a constant $\varepsilon = \varepsilon(m)$ such that the following estimate is true

$$K_B \leqslant \int |x_B(\omega)| \, d\mu_B^0 \leqslant C\varepsilon^{|B|-1} \tag{3.8}$$

and $\varepsilon(m) \to 0$ as $m \to 0$.

Lemma 3.2. With the same assumptions as in Lemma 3.1 there exists a constant c which does not depend on B such that

$$F_\Lambda(B) = \frac{Z_\beta(\Lambda \backslash B)}{Z_\beta(\Lambda)} \leqslant e^{c\,|B|}$$

and

$$\lim_{\Lambda \nearrow \mathbb{Z}^\nu} F_\Lambda(B) = F(B)$$

The proof of these lemmas is the content of the following section.

Proof of Theorem 2.2. Using (3.8) and the first inequality of Lemma 3.2 and taking into account that all sets $B_1,..., B_m$ in (3.7) are 1-connected, it is easy to get that

$$\sum_{B \subseteq \Lambda} |K_B(\mathscr{A}_{\mathscr{D}})| \, F_\Lambda(B) < R(\mathscr{D}) \cdot \max_{\omega} |\mathscr{A}_{\mathscr{D}}(\omega)| \tag{3.9}$$

where $R(\mathscr{D})$ is a constant independent of Λ. From this fact and the second inequality of Lemma 3.2, we can prove the existence of the following limit

$$\lim_{\Lambda \nearrow \mathbb{Z}^\nu} \sum_{B \subset \Lambda} K_B(\mathscr{A}_{\mathscr{D}}) \, F_\Lambda(B) \equiv \langle \mathscr{A}_{\mathscr{D}} \rangle$$

with the same bound as in (3.9).

Hence, there exists a probability measure $\mu_{\beta, \mathscr{D}}$ such that

$$\langle \mathscr{A}_{\mathscr{D}} \rangle = \int \mathscr{A}_{\mathscr{D}}(\omega) \, d\mu_{\beta, \mathscr{D}}$$

All these measures are compatible at $\mathscr{D}_1 \subset \mathscr{D}_2$ and consequently are generated by a unique limit measure μ_β on the σ-algebra Σ. ∎

4. CONVERGENCE OF THE CLUSTER EXPANSION

To prove the convergence of the cluster expansion (3.6) as $\Lambda \nearrow \mathbb{Z}^\nu$ we should prove Lemmas 3.1 and 3.2.

Proof of Lemma 3.1. First, let us note that because $0 \leqslant s_l \leqslant 1$, the expression in the curly brackets in the exponent in (3.3) can be estimated as follows

$$-\nu\lambda \sum_{j \in b} \int_0^\beta \omega_j^2(\tau) \, d\tau + \lambda \sum_{1 \leqslant l < m \leqslant n} s_l \cdots s_{m-1} \int_0^\beta \omega_{k_l}(\tau) \, \omega_{k_m}(\tau) \, d\tau$$
$$\leqslant -\frac{\lambda}{2} \sum_{\langle j, k \rangle \subset B} \int_0^\beta d\tau (|\omega_j(\tau)| - |\omega_k(\tau)|)^2 \leqslant 0$$

Thus we estimate this exponent by the unity.

Then, we apply Schwarz inequality with respect to the measure $d\mu^0_{\beta, B}$ in the expression for $K_B(\mathscr{A}_{\mathscr{D}})$ and taking into account that

$$\int d\mu^0_{\beta, \beta}(\omega_B)\, |\mathscr{A}(\omega_{k_0})|^2 = C^2_{k_0}(\mathscr{A}) = C^2_0 \tag{4.1}$$

we get

$$\int |x_B(\omega_B)|\, d\mu^0_{\beta, B} < C_0 \lambda^{n-1} \sum_\eta \int_0^1 (ds)^{n-1}\, h_\eta(s)\, J^{1/2}_B(\eta) \tag{4.2}$$

where

$$J_B(\eta) = \int d\mu^0_{\beta, B}(\omega_B) \prod_{l=2}^n \int_0^\beta d\tau_{\eta(l), l} \int_0^\beta d\tau'_{\eta(l), l}\, \omega_{k_{\eta(l)}}(\tau_{\eta(l), l})$$
$$\times\, \omega_{k_l}(\tau_{\eta(l), l})\, \omega_{k_{\eta(l)}}(\tau'_{\eta(l), l})\, \omega_{k_l}(\tau'_{\eta(l), l}) \tag{4.3}$$

Note that every η generates the graph $\tau(\eta)$ with vertices B and edges $\{k_l, k_{\eta(l)}\}$ which evidently constitutes a tree.

For a given η we denote by $d_\eta(l)$ the number of vertices $\{k_m\}$ in the graph $\tau(\eta)$ for which $\eta(m) = l$ (see ref. 14) and set

$$m_\eta(l) = \begin{cases} d_\eta(l), & \text{for } l = 1 \\ d_\eta(l) + 1, & \text{for } l \geqslant 2 \end{cases} \tag{4.4}$$

It is easy to see from the construction of η (see ref. 14) that

$$\sum_{l=1}^n d_\eta(l) = n - 1, \qquad |\eta| = n$$

To make the situation clear, consider the following example (Fig. 1), which corresponds to the expression (3.3) with $\eta(2) = 1$, $\eta(3) = \eta(4) = 2$ and $\eta(5) = 1$.

Note that every vertex (circle) of the graph corresponds to some site of the lattice, and every point (end of line) in a circle k_l corresponds to some $\omega_{k_l}(\tau_m)$.

Using the fact that $d\mu^0_{\beta, B}$ is a product measure and changing integrals with respect to $d\tau$ and $d\mu$, we rewrite $J_B(\eta)$ in (4.3) as follows:

$$J_B(\eta) = \prod_{l=2}^n \int_0^\beta d\tau_{\eta(l), l} \int_0^\beta d\tau'_{\eta(l), l} \prod_{p=1}^n \int d\mu^0_\beta(\omega_{k_p})\, \omega_{k_p}(\tau^{(1)}_p) \cdots \omega_{k_p}(\tau^{(2m_p)}_p) \tag{4.5}$$

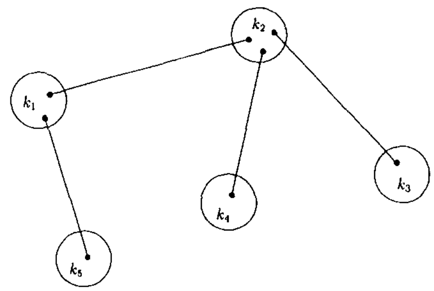

Figure 1

where $\tau_1^{(k)}$, $k = 1,\dots, 2m_1$, takes values in the set

$$\{\tau_{1,\,p_1}, \tau'_{1,\,p'_1} \mid p_1, p'_1 \in \eta^{-1}(1)\},$$

and

$$\eta^{-1}(1) = \{l \mid \eta(l) = 1\}.$$

For the graph in Fig. 1 we have $\eta^{-1}(1) = \{2, 5\}$, $\eta^{-1}(2) = \{3, 4\}$, $\eta^{-1}(3) = \eta^{-1}(4) = \eta^{-1}(5) = \varnothing$.

For $p \geqslant 2$, $\tau_p^{(k)}$, $k = 1,\dots, 2m_l$, takes values in the set

$$\{\tau_{\eta(l),\,l}, \tau'_{\eta(l),\,l}, \tau_{l,\,p_l}, \tau'_{l,\,p'_l} \mid p_l, p'_l \in \eta^{-1}(l)\}.$$

The integrals in the product of (4) define $2m_p$-time-point Green functions of the one-particle Hamiltonian (i.e., the Hamiltonian for one site k_p):

$$S_{2m_p}^{(k_p)}(\tau_p^{(1)},\dots, \tau_p^{(2m_p)}) \tag{4.6}$$

The assumptions on the interaction potential V allow to apply a Gaussian upper-bound inequality, which in our case is just a special case ($|\Lambda| = 1$, $\Lambda \equiv k_p$) of the Lemma 2.1 of ref. 5. Then

$$S^{(k_p)}{}_{2m_p}(\tau_p^{(1)},\dots, \tau_p^{(2m_p)}) \leqslant \sum_{\pi_p \in \mathscr{P}(2m_p)} \prod_{l=1}^{m_p} S_2^{(k_p)}(\tau_p^{\tau_p(2l-1)}, \tau_p^{\tau_p(2l)}) \tag{4.7}$$

where the sum over $\pi_p \in \mathscr{P}(2m_p)$ is the sum over different partitions indices $1,\dots, 2m_p$ into pairs; it consists of $(2mp)!\,(2^{m_p} m_p!)^{-1} \leqslant 2^{m_p} m_p!$ terms. Then, we can write for $J_B(\eta)$ the following inequality:

$$J_B(\eta) \leqslant \sum_{\pi = \{\pi_1,\dots, \pi_n\}} J_B^{(\pi)}(\eta) \tag{4.8}$$

where

$$J_B^{(\pi)}(\eta) = \int_0^{\hat\beta} d\tau_{\eta(2),2} \int_0^{\hat\beta} d\tau'_{\eta(2),2} \cdots \int_0^{\hat\beta} d\tau_{\eta(n),n} \int_0^{\hat\beta} d\tau'_{\eta(n),n} \prod_{p=1}^{n} \prod_{l=1}^{m_p} S_2(\tau_p^{\pi_p(2l-1)}, \tau_p^{\pi_p(2l)}) \tag{4.9}$$

We have dropped the indices (k_p) of the Green functions because they are the same for every site.

Note that $S_2(\tau_1, \tau_2) = s_2(\tau_1 - \tau_2)$, where $s_2(u)$ is a periodic function with period $\hat\beta$.

The next step is a trivial identity which follows from the construction of the graph $\tau(\eta)$ taking into account the particular partition π.

For given η, we consider $\tilde\tau(\eta)$, which can be constructed from η by doubling every rib (with its ends) of η (this is the result of applying Schwarz inequality). So, the graph $\tilde\tau(\eta)$ has the same form as in Fig. 1 but with a doubled number of ribs. Now. after applying a Gaussian upper-bound inequality, we should make pairings π_p, $p = 1,..., n$, of the points in every circle. Finally, we get the graph $\tau(\eta, \pi)$. See, for example, Fig. 2 with the graph $\tau(\eta, \pi)$ corresponding to the one from Fig. 1 (with some fixed pairings).

We denote every pairing by a bold line, and its analytic contribution is the two-point Green function $S_2(\tau_1, \tau_2)$.

Now, consider some graph $\tau(\eta, \pi)$ (with fixed pairings). Starting from some $k_p \in b$ along any rib (k_p, k_l) which connects k_p and k_l, we move until we reach the vertex k_l. Then, moving along the corresponding bold line in the circle k_l, we get to another rib (k_l, k_s) and, later, to the vertex k_s.

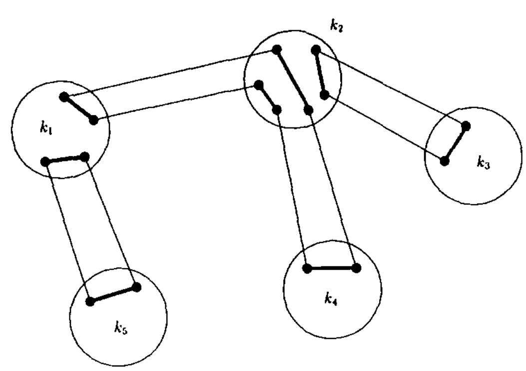

Figure 2

Repeating this process, we every time will move to the new rib until we get to the vertex k_p. It is easy to see that the graph $\tau(\eta, \pi)$ can be represented as the union of such closed paths with mutually non-coincident ribs. Let us put every such closed path of length $2p$ in correspondence with the following integral:

$$C_{2p} = \int_0^\beta d\tau_1 \cdots \int_0^\beta d\tau_{2p} s_2(\tau_1 - \tau_2)\, s_2(\tau_2 - \tau_3) \cdots s_2(\tau_{2p} - \tau_1) \tag{4.10}$$

Then, the following holds:

Proposition 4.1.

$$J_B^\pi(\eta) = \prod_{p=1}^{n-1} C_{2p}^{K_p} \tag{4.11}$$

where $K_p = K_p(\eta, \pi)$ is the number of closed paths of length $2p$ in the graph $\tilde{\tau}(\eta, \pi)$, and

$$0 \leqslant K_p \leqslant n-1, \qquad \sum_{p=1}^{n-1} pK_p = n-1 \tag{4.12}$$

Proof. Note that for every rib $b = (\eta(l), l)$ of the graph $\tau(\eta, \pi)$, we have to write in (4.9) the corresponding variables $\tau_{\eta(l),l} \equiv \tau_b$ and $\tau'_{\eta(l),l} \equiv \tau'_b$. So,

$$J_B^\pi = J_B(\tau(\eta, \pi)) = \int \prod_{b \in \tau(\eta, \pi)} d\tau_b \prod_{p=1}^{n} \prod_{\langle b, b' \rangle_p} S_2(\tau_b, \tau_{b'}) \tag{4.13}$$

where the product $\prod_{\langle b, b' \rangle_p}$ is over all pairings in the circle k_p.

Put in correspondence to every closed path $m = (b_1,..., b_{2k}) \subset \tau(\eta, \pi)$ the following product:

$$I_m(\tau_{b_1},..., \tau_{b_{2k}}) = S_2(\tau_{b_1}, \tau_{b_2})\, S_2(\tau_{b_2}, \tau_{b_3}) \cdots S_2(\tau_{b_{2k}}, \tau_{b_1}) \tag{4.14}$$

As for different closed paths m of the graph $\tau(\eta, \pi)$ the variables τ_b are different, we get that the integral (4.13) can be represented in the form of the following product:

$$J_B(\tau(\eta, \pi)) = \prod_{m \in \tau(\eta, \pi)} \int I_m(\tau_{b_1},..., \tau_{b_{2k}}) \prod_{b \in m} d\tau_b$$

Hence, taking into account the notation (4.10), we get (4.11). Then, (4.12) follows from the fact that $\sum_p (2k) \cdot K_p$ is exactly the number of ribs of the tree graph η. ▮

Now, taking into account (4.8) and (4.11), we obtain from (4.2) that

$$\int |x_B(\omega)| \, d\mu^0_{\beta, B} \leqslant c_0 \lambda^{n-1} \sum_\eta N_\eta^{1/2} \max_\pi \left\{ \int (ds)^{n-1} \, h_\eta(s) \prod_{p=1}^{n-1} C_{2pK_p/2} \right\} \tag{4.15}$$

where N_η is the number of terms in (4.8).

To estimate the last product in (4.15) let us write the Fourier transform for the periodic function $s_2(\tau - \tau')$:

$$s_2(\tau - \tau') = \frac{1}{\beta} \sum_{k \in \mathbb{Z}} \tilde{s}_2(k) \exp\left(i \frac{2\pi}{\beta} k(\tau - \tau') \right)$$

where

$$s_2(k) = \int_0^\beta s_2(u) \, e^{i(2\pi/\beta)} \, uk \, du \tag{4.16}$$

Let us use the following representation for $s_2(\tau_2 - \tau_1)$, $0 \leqslant \tau_1 < \tau_2 = \beta$:

$$s_2(\tau_2 - \tau_1) = \frac{\int_{\mathbb{R}} \int_{\mathbb{R}} y_2 P_{\tau_2 - \tau_1}(y_2/y_1) \, y_1 P_{\beta - (\tau_2 - \tau_1)}(y_1/y_2) \, dy_1 \, dy_2}{\int_{\mathbb{R}} P_\beta(y/y) \, dy} \tag{4.17}$$

where

$$P_\tau(z_1/z_2) = \frac{e^{-\tau \hat{h}}(z_1, z_2) \, \psi_0(z_1)}{e^{-E_0 \tau} \psi_0(z_2)} \tag{4.18}$$

is the density of the transition probabilities of the stationary Markov process ξ_t, $t \in \mathbb{R}^1$, which is generated by the one-particle Hamiltonian $\hat{h}$ (2.6) (see ref. 6 for details), ψ_0 is the ground state of the Hamiltonian $\hat{h}$, and E_0 is its eigenvalue. Then

$$e^{-\tau \hat{h}}(z_1, z_2) = \sum_{s=0}^{\infty} \psi_s(z_1) \, \psi_s(z_2) \, e^{-\tau E_s} \tag{4.19}$$

is the kernel of operator $e^{-\tau \hat{h}}$, ψ_s is the normalized eigenvector of $\hat{h}$, and E_s is the corresponding eigenvalue.

Inserting (4.17)–(4.19) in (4.16), we get

$$\tilde{s}_2(k) = \frac{1}{Z(\hat{\beta})} \sum_{s,\, s',\, s \neq s'} (x_{s,\, s'})^2 \frac{(E_s - E_{s'})(e^{-\hat{\beta} E_{s'}} - e^{-\hat{\beta} E_s})}{(E_s - E_{s'})^2 + 4\pi^2 k^2/\hat{\beta}^2} \tag{4.20}$$

where

$$Z(\hat{\beta}) = \sum_s e^{-\hat{\beta} E_s}$$

and

$$x_{ss'} = \int \psi_s(y)\, y \psi_{s'}(y)\, dy$$

($x_{ss} = 0$ because of condition (2.2)).

Thus

$$\tilde{s}_2(k) \leqslant \tilde{s}_2(0) \leqslant \frac{1}{\Delta^2} \tag{4.21}$$

where

$$\Delta = \min_{s \neq s'} |E_{s'} - E_s|$$

The inequality (4.21) follows from (4.20) with the using of the following direct calculations:

$$\frac{1}{Z(\hat{\beta})} \sum_{s,\, s'} (x_{ss'})^2 (E_s - E_{s'})(e^{-\hat{\beta} E_{s'}} - e^{-\hat{\beta} E_s}) = \frac{1}{Z(\hat{\beta})} \operatorname{Tr}([x, [\hat{h}, x]]\, e^{-\hat{\beta} \hat{h}}) = 1$$

The mass of the particle m is contained in the one-particle Hamiltonian $\hat{h}$ only through the expression for $\hat{V}$ (see (2.7)) and furthermore with positive powers we always can choose some constant C_1, which depends only on the coefficients a_{2p}, $p = 1,\ldots, s$ of the polynomial $V(q)$ and m_0' such that for all $0 \leqslant m < m_0'$

$$\frac{1}{\Delta^2} \leqslant c_1 \tag{4.22}$$

Then, from (4.10), (4.21) and (4.22) we get

$$C_{2p} \leqslant c_1^{2p-1} \sum_{k \in \mathbb{Z}} \tilde{S}_2(k) = C_1^{2p-1} \hat{\beta} S_2(0) = C_1^{2p-1} \hat{\beta} \langle x^2 \rangle \tag{4.23}$$

where

$$\langle x^2\rangle = \frac{\sum_s \int y^2 \psi_s \psi_s \, dy}{Z(\beta)} e^{-\beta E_s}$$

To estimate $\langle x^2\rangle$, we use the following inequality, see (ref. 15, Lemma 5.1).

$$\tilde{S}_2(0) \geqslant \hat{\beta}\langle x^2\rangle \, f\left(\frac{\hat{\beta}}{4\langle x^2\rangle}\right) \tag{4.24}$$

which follows from Theorem 3.1 of ref. 16. The function $f(\cdots)$ in (4.24) is defined by the relation[16]:

$$f(t \tanh t) = t^{-1} \tanh t, \qquad t \in (0, \infty).$$

We remember also that for our case $\beta \to \hat{\beta}$ and $m \to 1$. Then, we use the following inequality (see ref. 16 for details)

$$f(t) \geqslant t^{-1}(1 - e^{-t}) \tag{4.25}$$

obtaining from (4.21), (4.22), (4.24) and (4.25)

$$\langle x^2\rangle \leqslant \frac{\sqrt{C_1}}{2}\left(1 - \exp\left\{-\frac{\hat{\beta}}{4\langle x^2\rangle}\right\}\right)^{-1/2} \tag{4.26}$$

It is easy to see (for example, from a graphical consideration) that for any fixed β and sufficiently small $m_0 = m_0(\beta_0) < m_0'$ we can choose some constant C_2, which does not depend on β, such that for all $0 < m \leqslant m_0$

$$\langle x^2\rangle < C_2 \tag{4.27}$$

Now, it is enough for convergence to use the very rough estimate on N_η:

$$\begin{aligned}\sum_{\pi_1,\ldots,\pi_n} 1 = \prod_{p=1}^{n} \sum_{\pi_p} 1 = \prod_{p=1}^{n} \frac{(2m_p)!}{2^{m_p} m_p!} \\ \leqslant \prod_{p=1}^{n} \frac{(2d_\eta(p)+2)!}{2^{d_\eta(p)+1}(d_\eta(p)+1)!} \leqslant \prod_{p=1}^{n} (d_\eta(p)!)^2\end{aligned} \tag{4.28}$$

As a result we obtain from (2.8), (4.12), (4.13), (4.23), (4.27) and (4.28)

$$|K_B(\mathscr{A}_{\mathscr{D}})| \leqslant c_0(\lambda\beta^{1/2}m^{-s/2(s+1)}C_1C_3)^{n-1}\sum_{\eta}\prod_{p=1}^{n} d_\eta(p)!\int_0^1 (ds)^{n-1}\,h_\eta(s)$$

where $C_3 := \max\{1, (C_2/C_1)^{1/2}\}$.

The last step of our proof consists in the use of the Battle–Federbush inequality[14, 17]:

$$\sum_{\eta}\prod_{p=1}^{n} d_\eta(p)!\int_0^1 (ds)^{n-1}\,h_\eta(s) \leqslant 4^n, \qquad n=|\eta|$$

This yields (3.8) with

$$C = C_0(\lambda\beta^{1/2}m^{-s/2(s+1)}C_1C_3)^{-1}$$

and

$$\varepsilon = 4\pi\beta^{1/2}m^{-s/2(s+1)}C_1C_3 = 4JC_1C_3m^{(s-2)/2(s+1)}\beta^{1/2}. \quad \blacksquare$$

Proof of Lemma 3.2. To prove Lemma 3.2, we can use the general theory of polymer-type expansions[18] (or, equivalently, Kirkwood–Salsburg type considerations[19–22]), which is based on the following cluster estimate:

$$\sup_t \sum_{\substack{B\subset\mathbb{Z}^\nu\\ t\in B,\,|B|=n}} \frac{|K_B(\mathbf{1})|}{\prod_{j\in b}K_{\{j\}}} < \text{const}\cdot\varepsilon_0^n, \qquad |B|=n>1 \tag{4.29}$$

where ε is sufficiently small.

In turn, (4.29) follows from Lemma 3.1, the fact that the number of sets B with a fixed site $\{t\}$ and $|B|=n$ is less than $(2^\nu)^n$, and the following proposition:

Proposition 4.2. For the one-particle interaction (2.6)–(2.7), we can choose a sufficiently small mass such that

$$K_t = \int \exp\left\{-\lambda v\int_{S_\beta}\omega_t^2(\tau)\,d\tau\right\} d\mu_{\beta,t}(\omega_t) > \tfrac{1}{2} \tag{4.30}$$

The proof of this statement easily follows from the asymptotic limit

$$\lim_{m \to 0} \int \exp\left\{-\lambda v \int_{S_\beta} \omega_t^2(\tau)\, d\tau\right\} d\mu^0_{\beta,t}(\omega_t) = \lim_{m \to 0} \frac{\operatorname{Tr} e^{-\hat{\beta}(\hat{h}_t + \lambda v x_t^2)}}{\operatorname{Tr} e^{-\hat{\beta}\hat{h}_t}} = 1$$

if we take into account that at $m < 0$, $\hat{\beta} \to \infty$ and $\lambda \to 0$ (see (2.8) and (3.2)) and the fact that the operators $\hat{h}_t$ (see (2.6), (2.7)) and $\hat{h}_t + \lambda v x_t^2$ have the same ground states in the limit $m \to 0$. ∎

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